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Elgenvalues of Fibonacci-like Sequences

Elyssa Hurst
Western Kentucky University

Eigenvalues of

Fibonacci-like Sequences

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Elyssa Hurst

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Mark P. Robinson

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Directed by: Randall Swift, Daniel Biles, and Mark Robinson

Department of Mathematics Western Kentucky University

The familiar Fibonacci sequence

$$1,1,2,3,5,8,13,\ldots$$

can be described by the recurrence relation

$$x(0) = 1,$$

$$x(1) = 1,$$

$$x(n) = x(n-1) + x(n-2).$$

For this relation, as $n \to \infty$,

$$\frac{x(n+1)}{x(n)} \to \frac{1+\sqrt{5}}{2},$$

which is the familiar golden ratio. This value is also the dominant eigenvalue of the above recurrence relation. In this series, we consider the dominant eigenvalue of some Fibonacci-like sequence of the form

$$x(n) = \sum_{k=1}^{n-1} a_k Z_k x(n-k)$$

where the Z_k's are independent random variables with

$$Z_k = \begin{cases} +1 & \text{with probability } p, \\ -1 & \text{with probability } q, \end{cases}$$

with $p + q = 1$, and for each k, the a_k's are either 0 or 1.

Chapter One

A Brief History of Fibonacci Sequences

■ 1.1 Introduction

It was early in the thirteenth century that Leonardo of Pisa, more commonly known as Fibonacci, studied the "rabbit problem." This problem was first introduced in his book entitled *Liber Abaci*, published in 1202, which explored the use of Arabic numerals [9]. The "rabbit problem" brings about the sequence of numbers familiarly known as the Fibonacci numbers.

It started with a single pair of rabbits in a confined area. This pair, and each subsequent pair, would produce a new pair each month, once they have reached two months old. The question then that Fibonacci asked was, "How many pairs of rabbits will be present after just one year?" It is assumed that no deaths occur within this population. So, in the first month there is the original pair of rabbits. The second month still has only that single pair, and finally in the third month there is a new pair. Continuing in this fashion, we are looking at the following:

$$F_{n+2} = F_{n+1} + F_n,$$

where F_n and F_{n+1} are the number of pairs present at the n^{th} and $(n + 1)^{th}$ month. Thus, the sum of pairs at the n^{th} and $(n + 1)^{th}$ months yield the total pairs present at the $(n + 2)^{th}$ month. This recurrence relation generates the following sequence:

n	1	2	3	4	5	6	7	...
F_n	1	1	2	3	5	8	13	...

As this sequence progresses, we discover the answer to Fibonacci's question is that there will be 144 pairs of rabbits present at the end of the year [7]. This sequence is widely known as the Fibonacci sequence, and its individual terms as Fibonacci numbers.

These Fibonacci numbers occur quite often in our natural surroundings. For instance, consider the number of petals found on a daisy, there are usually 13, 21, or 34. Sunflowers are another perfect example. Their seeds spiral out from the center, some in one direction and the rest in another. The number of spirals going in each direction are adjacent Fibonacci numbers. The Fibonacci numbers also appear in the foods we eat, for a bell pepper has 3 chambers and an apple has a 5-point-star cross section. The Fibonacci sequence is also apparent when observing the genealogy of a drone, male bee [4].

Not only does this sequence of numbers appear in nature; but it also comes up frequently within the realm of mathematics; that is, many geometric interpretations involve the Fibonacci numbers or at least they make their appearance. Through the use of infinite simple continued fractions, it is observed that the equation

$$\lambda^2 - \lambda - 1 = 0$$

has the positive solution

$$\lambda = \frac{1+\sqrt{5}}{2} \approx 1.61.$$

This positive λ value is known as the *Golden Mean,* or sometimes referred to as the *Golden Ratio* [5].

Indeed, the Fibonacci numbers form an intriguing sequence that seems to take shape in practically every aspect of our lives, whether we notice or not. In fact, there is even a Mathematics Research journal, *The Fibonacci Quarterly*, that focuses solely on this sequence and its applications.

Chapter Two

Fibonacci Sequences

■ 2.1 The Classic Sequence

The familiar Fibonacci sequence is

$$1,1,2,3,5,8,13,\ldots,$$

where each successive term of the sequence is obtained by adding the previous two terms. One way to view this sequence is by looking at the recurrence relation,

$$x(n) = x(n-1) + x(n-2),$$

where $x(n)$ is the n^{th} term in the sequence. When looking at the classic Fibonacci case, the first two terms of the sequence are 1 and 1. Thus $x(0) = 1$ and $x(1) = 1$.

Given the recurrence relation,

$$x(n) = x(n-1) + x(n-2)$$

for $n \geq 2$ and the first two terms $x(0) = 1$ and $x(1) = 1$, we can solve the recurrence relation to obtain a closed form expression. We can write

$$x(0) = 1,$$

$$x(1) = 1,$$

$$x(n) = x(n-1) + x(n-2).$$

Now, subtracting x(n-1) and x(n-2) from both sides gives the difference equation:

$$x(n) - x(n-1) - x(n-2) = 0. \tag{1}$$

The Fibonacci sequence exhibits growth in magnitude, so if we let

$$x(n) = c\lambda^{n},$$

the above equation can be written in the form

$$c\lambda^{n} - c\lambda^{n-1} - c\lambda^{n-2} = 0. \tag{2}$$

Equation (2) is known as the *characteristic equation* for the above difference equation (1). Each term in equation (2) possesses a common factor of $c\lambda^{n-2}$, which can be factored out, giving the equation

$$c\lambda^{n-2}(\lambda^{2} - \lambda - 1) = 0.$$

With this particular equation and by way of the Principle of Zero Products,

$$c\lambda^{n-2} = 0 \tag{3}$$

or

$$\lambda^2 - \lambda - 1 = 0. \tag{4}$$

Equation (3) is uninteresting, since solving for λ yields $\lambda=0$. Thus, we are interested in equation (4) and the λ values it will yield.

Here we have a quadratic equation; solving for λ gives

$$\lambda = \frac{-(-1)\pm\sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{1+4}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2}.$$

These values, $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$, are the roots of the characteristic equation and are commonly known as *eigenvalues.* Since $\frac{1+\sqrt{5}}{2}$ is the larger of the two values, it is often referred to as the *dominant eigenvalue.* We have seen this value before; it is the *Golden Ratio.* Once the eigenvalues of a recurrence relation have been obtained from the characteristic equation, they can be used to find a closed form for the solution of the recurrence relation. In fact, we have the following theorem.

Theorem 2.1: If a characteristic equation has m distinct roots, λ_1, λ_2, λ_3,..., λ_m, then $x(n) = c_1\lambda_1{}^n + c_2\lambda_2{}^n + c_3\,\lambda_3{}^n + \ldots + c_m\lambda_m{}^n$, where c_1, c_2, c_3,..., c_m are constants.

For further analysis, see Mooney and Swift *A Course in Mathematical Modeling* [2]. In the case of the Fibonacci relation, the eigenvalues are $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$. So, to obtain a closed form for the recurrence relation

$$x(0) = 1,$$

$$x(1) = 1,$$

$$x(n) = c_1\lambda_1{}^n + c_2\lambda_2{}^n,$$

we have

$$x(n) = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + \quad c_2\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

So if $x(0) = 1$, then $c_1 + c_2 = 1$, so $c_1 = 1 - c_2$.

Now, we can write $x(n)$ in terms of only c_2, thus

$$x(n) = (1 - c_2)\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

So, if $x(1) = 1$, then

$$(1 - c_2)\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = 1.$$

Thus,

$$\frac{\sqrt{5}-1}{2} = c_2\sqrt{5},$$

and

$$\frac{\sqrt{5}-1}{2\sqrt{5}} = c_2.$$

Rationalizing the denominator gives $c_2=\frac{5-\sqrt{5}}{10}$ and using $c_1=1$ - c_2 gives

$$c_1 = \frac{10}{10} - \frac{5-\sqrt{5}}{10} = \frac{10-5+\sqrt{5}}{10} = \frac{5+\sqrt{5}}{10}.$$

Now that c_1 and c_2 are known, the equation

$$x(n) = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

becomes

$$x(n) = \left(\frac{5+\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{5-\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n. \quad (5)$$

This closed form expression for the Fibonacci relation is rather interesting. Note that each term is the product of irrational numbers, whereas the Fibonacci sequence is integer valued.

Once we have obtained a closed form expression for a recurrence relation, our analysis becomes simpler; for instance, as n approaches infinity, we can show that the ratio $\frac{x(n+1)}{x(n)}$ converges to the *Golden Ratio*, $\frac{1+\sqrt{5}}{2} \approx 1.61$.

To show this, consider the following:

$$\lim_{n\to\infty} \frac{x(n+1)}{x(n)}$$
$$= \lim_{n\to\infty} \frac{\left(\frac{5+\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + \left(\frac{5-\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\left(\frac{5+\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n} + \left(\frac{5-\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{n}}.$$

Since $\left|\frac{1-\sqrt{5}}{2}\right| < 1$, the terms $\left(\frac{1-\sqrt{5}}{2}\right)^n$ and $\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$ go to zero as n goes to infinity; that is,

$$\lim_{n\to\infty} \frac{x(n+1)}{x(n)}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{5+\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}}{\left(\frac{5+\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{n}}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}}{\left(\frac{1+\sqrt{5}}{2}\right)^{n}} = \frac{1+\sqrt{5}}{2}.$$

Therefore, $\lim_{n\to\infty} \frac{x(n+1)}{x(n)} = \frac{1+\sqrt{5}}{2}$. The expression $\frac{x(n+1)}{x(n)}$ represents the growth of the relation x(n). So, the Golden Ratio is the growth rate of the Fibonacci sequence.

An interesting question is, will this result still hold when x(0) and x(1) are not equal to 1; that is, is the long term growth rate of the sequence dependent upon its initial values?

Consider the following example,

$$x(0) = 3,$$

$$x(1) = 4,$$

so that the Fibonacci sequence now looks like 3, 4, 7, 11, 18, 29, ... The recurrence relation x(n) = x(n-1) + x(n-2) still holds where x(n) is the n^{th} term of the sequence, obtained by adding the previous two terms. So, now we can solve the given recurrence relation to obtain a closed form.

$$x(0) = 3,$$

$$x(1) = 4,$$

$$x(n) = x(n-1) + x(n-2).$$

This recurrence relation has characteristic equation,

$$c\lambda^{n} - c\lambda^{n-1} - c\lambda^{n-2} = 0,$$

which has roots $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$.

It is not until after finding the λ's and beginning to solve for the closed form of the recurrence relation that we notice any differences the x(0) = 3 and x(1) = 4 introduce.

Since

$$x(n) = c_1\lambda_1^{\,n} + c_2\lambda_2^{\,n},$$

$$x(0) = 3,\ \ x(1) = 4,$$

we have

$$x(n) = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

If x(0) = 3, then $c_1 + c_2 = 3$, so that

$$c_1 = 3 - c_2.$$

Now we can rewrite x(n) in terms of only c_2, thus

$$x(n) = (3 - c_2)\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

So, if x(1) = 4, then

$$(3 - c_2)\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = 4.$$

Thus,

$$c_2 = \frac{15-5\sqrt{5}}{10} = \frac{3-\sqrt{5}}{2},$$

and using $c_1 = 3 - c_2$ gives

$$c_1 = \frac{15+5\sqrt{5}}{10} = \frac{3+\sqrt{5}}{2}.$$

Now that c_1 and c_2 are known, the equation

$$\mathrm{x(n)} = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n,$$

becomes

$$\mathrm{x(n)} = \left(\frac{3+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

This closed form expression is different from the classic Fibonacci sequence; however, it can be shown that the ratio $\frac{x(n+1)}{x(n)}$ converges to the Golden Ratio, $\frac{1+\sqrt{5}}{2}$, by the same methods used to find the limit of the classic Fibonacci case.

From these examples, it is not hard to see that changing the initial values of the sequence does not affect the λ's (eigenvalues). In fact, the ratio $\frac{x(n+1)}{x(n)}$ still converges to the same Golden Ratio no matter what the initial values are. The only differences that arise are the values of c_1

and c_2 in the closed form solution.

2.2 Some Extensions

The classic Fibonacci sequence considers the sum of only two terms. This sequencing technique can also be implemented when looking at the sum of three terms, or indeed, even more terms. For instance, consider the sequence, 1,1,1,3,5,9,17,..., obtained when beginning with a "seed" of (1,1,1). This three-term sequencing has been coined the "Tribonacci" sequence, as mentioned in *Fibonacci and Lucas Numbers*, and we will follow this terminology [7].

The recurrence relation for this sequence can be written as

$$x(n) = x(n-1) + x(n-2) + x(n-3), \tag{6}$$

where x(n) is the n^{th} term in the sequence. In this Tribonacci case, the natural first three terms are 1, 1, and 1. Given this information the recurrence relation is

$$x(0) = 1,$$

$$x(1) = 1,$$

$$x(2) = 1,$$

$$x(n) = x(n-1) + x(n-2) + x(n-3).$$

The above equation can be rewritten as the difference equation

$$x(n) - x(n-1) - x(n-2) - x(n-3) = 0.$$

By letting $x(n) = c\lambda^n$ the equation can be written in the form

$$c\lambda^n - c\lambda^{n-1} - c\lambda^{n-2} - c\lambda^{n-3} = 0. \tag{7}$$

Equation (7) is the characteristic equation for the difference equation associated with the Tribonacci case. Each term of this characteristic equation has a common factor of $c\lambda^{n-3}$, which can be factored out, giving the equation

$$c\lambda^{n-3}(\lambda^3 - \lambda^2 - \lambda - 1) = 0.$$

By way of the Principle of Zero Products, we are interested in the equation

$$\lambda^3 - \lambda^2 - \lambda - 1 = 0.$$

To solve this cubic equation for λ, we shall employ the use of *Mathematica*. Typing the following command in *Mathematica* yields the three roots of the above equation.

n = 3;

$$\texttt{Solve}\left[x^n - \sum_{k=0}^{n-1} x^k == 0,\ x\right]$$

$$\{\{x \to \frac{1}{3} + \frac{1}{3}\left(19 - 3\sqrt{33}\right)^{1/3} + \frac{1}{3}\left(19 + 3\sqrt{33}\right)^{1/3}\},$$
$$\{x \to \frac{1}{3} - \frac{1}{6}\left(1 + I\sqrt{3}\right)\left(19 - 3\sqrt{33}\right)^{1/3} - \frac{1}{6}\left(1 - I\sqrt{3}\right)\left(19 + 3\sqrt{33}\right)^{1/3}\},$$
$$\{x \to \frac{1}{3} - \frac{1}{6}\left(1 - I\sqrt{3}\right)\left(19 - 3\sqrt{33}\right)^{1/3} - \frac{1}{6}\left(1 + I\sqrt{3}\right)\left(19 + 3\sqrt{33}\right)^{1/3}\}\}$$

The last two solutions for λ are complex, so we need to calculate the modulus of each of these solutions to determine the dominant eigenvalue. Thus, using *Mathematica,*

$$\textbf{NSolve}\left[x^n - \sum_{k=0}^{n-1} x^k == 0,\ x\right]$$

{{x→-0.419643-0.606291 I},{x→-0.419643+0.606291 I},{x→1.83929}}

Notice that the first two solutions generated are complex conjugates, thus having the same modulus, so we need to calculate the modulus of only one of them.

Abs[-0.419643 + 0.606291 I]

0.737353

Thus $\lambda \approx 1.83$ is the dominant eigenvalue. Now that we have the eigenvalues of the recurrence relation, we could obtain a closed form using the same method as before. However, since we have two complex solutions, finding the closed form of the recurrence relation is difficult.

Instead, we analyze the behavior of the dominant eigenvalue of the Tribonacci sequence. As noted, when looking at the classic Fibonacci case, as n approached infinity, the ratio $\frac{x(n+1)}{x(n)}$ approached the dominant eigenvalue. Thus, it would seem that as n approaches infinity in the Tribonacci case, $\frac{x(n+1)}{x(n)}$ would approach its relevant dominant eigenvalue. Therefore, $\frac{x(n+1)}{x(n)}$ should converge to 1.83. Let's show this in general; that is, we show that

$$\frac{x(n+1)}{x(n)} \to \lambda_{\text{dom}} \text{ as } n \to \infty,$$

where λ_{dom} is the dominant eigenvalue of the difference equation.

Consider the generalized linear difference equation:

$$a_0 x(n) + a_1 x(n-1) + a_2 x(n-2) + \ldots + a_{n-1} x(1) + a_n x(0) = 0.$$

If this equation has m distinct eigenvalues, $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_m$, then, by Theorem 2.1, the solution can be written in the closed form

$$x(n) = c_1\lambda_1^n + c_2\lambda_2^n + c_3\lambda_3^n + \ldots + c_m\lambda_m^n,$$

where $c_1, c_2, c_3, \ldots, c_m$ are constants and the λ's are written in increasing value. Thus,

$$|\lambda_1^n| < |\lambda_2^n| < |\lambda_3^n| < \ldots < |\lambda_m^n|.$$

As $n \to \infty$, we wish to show that $\lim_{n\to\infty} \frac{x(n+1)}{x(n)} \longrightarrow \lambda_{\text{dom}}$ (the dominant eigenvalue). So,

$$\lim_{n\to\infty} \frac{x(n+1)}{x(n)} = \lim_{n\to\infty} \frac{c_1\lambda_1^{n+1} + c_2\lambda_2^{n+1} + c_3\lambda_3^{n+1} + \ldots + c_m\lambda_m^{n+1}}{c_1\lambda_1^n + c_2\lambda_2^n + c_3\lambda_3^n + \ldots + c_m\lambda_m^n}.$$

Since λ_m is the largest eigenvalue, we can write

$$\lim_{n\to\infty} \frac{c_1\lambda_1^{n+1} + c_2\lambda_2^{n+1} + c_3\lambda_3^{n+1} + \ldots + c_m\lambda_m^{n+1}}{c_1\lambda_1^{n} + c_2\lambda_2^{n} + c_3\lambda_3^{n} + \ldots + c_m\lambda_m^{n}} \times \frac{(\frac{1}{\lambda_m})^{n+1}}{(\frac{1}{\lambda_m})^{n}} \times \frac{1}{(\frac{1}{\lambda_m})},$$

which yields,

$$\lim_{n\to\infty} \frac{c_1(\frac{\lambda_1}{\lambda_m})^{n+1} + c_2(\frac{\lambda_2}{\lambda_m})^{n+1} + c_3(\frac{\lambda_3}{\lambda_m})^{n+1} + \ldots + c_m(\frac{\lambda_m}{\lambda_m})^{n+1}}{c_1(\frac{\lambda_1}{\lambda_m})^{n} + c_2(\frac{\lambda_2}{\lambda_m})^{n} + c_3(\frac{\lambda_3}{\lambda_m})^{n} + \ldots + c_m(\frac{\lambda_m}{\lambda_m})^{n}} \times \lambda_m$$

$$= \frac{0+0+0+\ldots+c_m(1)}{0+0+0+\ldots+c_m(1)} \times \lambda_m \quad = \lambda_m = \lambda_{\text{dom}}.$$

Thus, in the general case, the ratio $\frac{x(n+1)}{x(n)}$ converges to the dominant eigenvalue.

■ 2.3 Some Subsequences

We have already seen that in the classic Fibonacci case, the ratio $\frac{x(n+1)}{x(n)}$ converges to the approximate value 1.61, and in the Tribonacci case, it converges to the approximate value 1.83. These values, in each case, are the dominant eigenvalues. How do these eigenvalues compare to other situations proposed by the Fibonacci sequencing method?

For instance, let's examine the Tribonacci case, where we look back three terms, and all of its subcases. Consider choosing only two of the three terms that are being summed. There are only three subcases using this technique. They are

$$x(n) = x(n-1) + x(n-2) \tag{8}$$

$$x(n) = x(n-1) + x(n-3) \tag{9}$$

$$x(n) = x(n-2) + x(n-3). \tag{10}$$

Equation (8) is the recurrence relation for the classic Fibonacci sequence, which we know has an approximate dominant eigenvalue of 1.61. The dominant eigenvalues for equations (9) and (10) can be found using the NSolve command in *Mathematica*.

```
NSolve[x^2 - x - 1 == 0, x]
{{x→-0.618034},{x→1.61803}}
NSolve[x^3 - x^2 - 1 == 0, x]
{{x→-0.232786-0.792552 I},{x→-0.232786+0.792552 I},{x→1.46557}}
Abs[-0.232786 + 0.792552 I]
0.826031
NSolve[x^3 - x - 1 == 0, x]
{{x→-0.662359-0.56228 I},{x→-0.662359+0.56228 I},{x→1.32472}}
Abs[-0.662359 + 0.56228 I]
0.868837
```

Thus, equation (9) has dominant eigenvalue approximately 1.46, and similarly equation (10) has a dominant eigenvalue approximately 1.32.

Now,

$$1.32 < 1.46 < 1.61,$$

thus the dominant eigenvalue of (10) is the smallest and (8) is the largest. This suggests that looking back at the previous two successive terms yields a larger dominant eigenvalue than one previous and a further back term. This is evident, considering the n^{th} term in the series is the sum of the sequences of previous terms. Thus, as the sequence progresses the n^{th} terms become larger in magnitude. Hence, there is some k for which

$$|x(k)| \leq |x(k+1)| \leq |x(k+2)| \leq \ldots$$

for each sequence.

Extending the Fibonacci sequence to that created from four terms becomes tedious in its representation. Thus, the introduction of a compact notation proves beneficial.

Consider

$$x_{k,s}{}^{\{1,2,\ldots,k\}}(n),$$

where n is the current term within the summation, k is the total number of terms we are looking back, and s is the length of a particular subcase of the k terms. Thus, the superscript of $\{1,2,\ldots,k\}$ represents the specific terms we are considering in the summation and there are at most s terms represented.

For example, the three subcases determined in the Tribonacci case would be represented as follows:

$$x_{3,2}^{\{1,2\}}(n) = \text{x(n-1)} + \text{x(n-2)},$$

$$x_{3,2}^{\{1,3\}}(n) = \text{x(n-1)} + \text{x(n-3)},$$

$$x_{3,2}^{\{2,3\}}(n) = \text{x(n-2)} + \text{x(n-3)}.$$

Thus, the normal Fibonacci sequence on four terms is represented as:

$$x_{4,4}^{\{1,2,3,4\}}(n).$$

As in the Tribonacci case, we shall consider each of the subcases that arise when considering four terms. In this case, we can choose either two of the four terms, or three of the four terms. When considering only two of the four terms, there exist $\binom{4}{2}$ or six subcases. Similarly, when choosing three of the four terms, there are $\binom{4}{3}$ or four subcases -- making a total of ten subcases using this technique. The following table lists each of these ten subcases and their approximate dominant eigenvalues.

Choose 2 terms only:	Choose 3 terms only:
$x_{4,2}^{\{1,2\}}(n),\ \lambda_{\text{dom}} \approx 1.61$	$x_{4,3}^{\{1,2,3\}}(n),\ \lambda_{\text{dom}} \approx 1.83$
$x_{4,2}^{\{1,3\}}(n),\ \lambda_{\text{dom}} \approx 1.46$	$x_{4,3}^{\{1,2,4\}}(n),\ \lambda_{\text{dom}} \approx 1.75$
$x_{4,2}^{\{1,4\}}(n),\ \lambda_{\text{dom}} \approx 1.38$	$x_{4,3}^{\{1,3,4\}}(n),\ \lambda_{\text{dom}} \approx 1.61$

$x_{4,2}{}^{\{2,3\}}(n),\ \lambda_{\text{dom}} \approx 1.32$ $x_{4,3}{}^{\{2,3,4\}}(n),\ \lambda_{\text{dom}} \approx 1.46$

$x_{4,2}{}^{\{2,4\}}(n),\ \lambda_{\text{dom}} \approx 1.27$

$x_{4,2}{}^{\{3,4\}}(n),\ \lambda_{\text{dom}} \approx 1.22$

This table represents the dominant eigenvalue to which the ratio $\frac{x(n+1)}{x(n)}$ converges for each subcase derived from four terms. These approximate values were found by employing *Mathematica* and the NSolve command. As we noticed in the Tribonacci case, the recurrence relation for the classic Fibonacci case appears once again. Also, the general Tribonacci sequence and each of its subcases are found among the subcases of the four-term sequencing. This implies that each "n"bonacci sequence and all of its subcases contain every "m"bonacci sequence and each of its subcases, where $2 \leq m < n$.

To compare each of these relations, we shall introduce a similar notation as before, where

$$x_{k,s}{}^{\{1,2,\ldots k\}}(n) \text{ converges to } \lambda_{k,s}{}^{\{1,2,\ldots k\}},$$

and $\lambda_{k,s}{}^{\{1,2,\ldots k\}}$ is the dominant eigenvalue. For example,

$$x_{4,2}{}^{\{1,2\}}(n) \longrightarrow \lambda_{4,2}{}^{\{1,2\}}$$

$$x_{4,2}{}^{\{1,3\}}(n) \longrightarrow \lambda_{4,2}{}^{\{1,3\}}.$$

Comparing the dominant eigenvalues of the case considering only two terms yields the following:

$$\lambda_{4,2}{}^{\{3,4\}} < \lambda_{4,2}{}^{\{2,4\}} < \lambda_{4,2}{}^{\{2,3\}} < \lambda_{4,2}{}^{\{1,4\}} < \lambda_{4,2}{}^{\{1,3\}} < \lambda_{4,2}{}^{\{1,2\}}. \tag{11}$$

Also, comparing the dominant eigenvalues of the method considering three terms yields the following:

$$\lambda_{4,3}{}^{\{2,3,4\}} < \lambda_{4,3}{}^{\{1,3,4\}} < \lambda_{4,3}{}^{\{1,2,4\}} < \lambda_{4,3}{}^{\{1,2,3\}}. \tag{12}$$

Both of these inequalities, (11) and (12), concur with our earlier findings surrounding the Tribonacci sequence, whereas the dominant eigenvalue becomes larger the fewer further back terms that are considered.

■ 2.4 The Dominant Eigenvalue in the Limiting Case

As n increases, the dominant eigenvalue to which the sequence converges increases. This is evident from our previous work:

$$\lambda_{2,2}{}^{\{1,2\}} < \lambda_{3,3}{}^{\{1,2,3\}} < \lambda_{4,4}{}^{\{1,2,3,4\}}. \tag{13}$$

We know that each sequence converges to its dominant eigenvalue. So, ultimately, what does the ratio of "n"bonacci sequences, $n \geq 2$, converge to, as n increases?

Using *Mathematica*, it can be shown numerically that for n large, the sequence is converging to 2. Consider the following *Mathematica* command

```
Table[{n, Max[Abs[
    N[Table[Solve[x^n - Sum[x^k, {k, 0, n-1}] == 0, x] [[k]][[1]][[2]], {k, 1, n}], 8]]]},
  {n, 1, 30}] //
 TableForm
```

$$N\left[\mathrm{Table}\left[\mathrm{Solve}\left[x^n - \sum_{k=0}^{n-1} x^k == 0,\ x\right]\right]\right]$$

1	1.
2	1.618034
3	1.8392868
4	1.927562
5	1.9659482
6	1.9835828
7	1.9919642
8	1.9960312
9	1.9980295
10	1.9990186
11	1.9995104
12	1.9997555
13	1.9998778
14	1.9999389
15	1.9999695
16	1.9999847
17	1.9999924
18	1.9999962
19	1.9999981
20	1.999999
21	1.9999995
22	1.9999998
23	1.9999999
24	1.9999999
25	2.
26	2.
27	2.
28	2.
29	2.
30	2.

This table shows numerically that as n grows, $\lambda_{max} \approx 2$. Now we show it analytically. First consider f(x) defined by

$$f(x) = x^n - \sum_{k=0}^{n-1} x^k = x^n - x^{n-1} - \ldots - x - 1$$

now

$$f(1) = 1 - \sum_{k=0}^{n-1} 1 = 1 - n < 0 \text{ for } n > 1$$

and

$$f(2) = 2^n - \sum_{k=0}^{n-1} 2^k = 2^n - \left(\frac{1-2^n}{1-2}\right) = 2^n + 1 - 2^n = 1 > 0$$

Since f is a polynomial, it is continuous. Hence by the Intermediate Value Theorem, there is a root for $f(x) = x^n - \sum_{k=0}^{n-1} x^k$ that lies between 1 and 2.

In fact, we can say

$$\begin{aligned} f(1+\varepsilon) &= (1+\varepsilon)^n - \sum_{k=0}^{n-1} (1+\varepsilon)^k \\ &= (1+\varepsilon)^n - \frac{1-(1+\varepsilon)^n}{1-(1+\varepsilon)} \\ &= (1+\varepsilon)^n - \frac{1-(1+\varepsilon)^n}{\varepsilon} \\ &= \frac{(1+\varepsilon)^n \varepsilon + 1 - (1+\varepsilon)^n}{\varepsilon} \\ &= \frac{1 + (1+\varepsilon)^n (\varepsilon - 1)}{\varepsilon}. \end{aligned}$$

Choose $\varepsilon = .1$, so

$$f(1+.1) = \frac{1+(1.1)^n(-.9)}{.1}.$$

We want to consider the following and solve for n

$$1 + (1.1)^n(-.9) < 0$$

$$(1.1)^n > \frac{1}{.9}.$$

Here, we need to take the logarithm of both sides. So,

$$n\ln(1.1) > \ln(\tfrac{1}{.9})$$

$$n > \frac{\ln(\frac{1}{.9})}{\ln(1.1)}$$

thus,

$$n > 1.105.$$

So, as long as $n > 1.01$, the λ_{max} is not converging to 1.

Now, we wish to show $\lambda_{max} \to 2$, where λ_{max} is

$$\lambda_{max} = \max\{|\lambda_1|, |\lambda_2|, |\lambda_3|, ..., |\lambda_n|\}$$

and, $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$ are the roots of

$$\lambda^n - \lambda^{n-1} - \lambda^{n-2} - ... - \lambda - 1 = 0.$$

This equation resembles a geometric series, where we have $1 + r + r^2 + ... + r^n$, except in this case we are dealing with subtraction, yet nothing need change. Hence, we can use this information to rewrite

$$\lambda^n - \lambda^{n-1} - \lambda^{n-2} - \ldots - \lambda - 1 = 0.$$

First, consider

$$\lambda^{n-1} + \lambda^{n-2} + \ldots + \lambda + 1 = \sum_{k=0}^{n-1} \lambda^k = \frac{1-\lambda^n}{1-\lambda}.$$

Thus,

$$\lambda^n - \lambda^{n-1} - \lambda^{n-2} - \ldots - \lambda - 1 = 0,$$

$$\lambda^n - (\lambda^{n-1} + \lambda^{n-2} + \ldots + \lambda + 1) = 0,$$

which is

$$\lambda^n - \sum_{k=0}^{n-1} \lambda^k = 0,$$

implying

$$\lambda^n - \frac{1-\lambda^n}{1-\lambda} = 0. \tag{14}$$

We are trying to find λ_{max}, so if we solve this equation in terms of λ, we have

$$\frac{\lambda^n(1-\lambda)}{1-\lambda} - \frac{1-\lambda^n}{1-\lambda} = 0,$$

which gives

$$\frac{\lambda^n(1-\lambda) - (1-\lambda^n)}{1-\lambda} = 0.$$

Some algebra yields

$$\frac{-\lambda^{n+1} - 1 + 2\lambda^n}{1-\lambda} = 0.$$

Thus,

$$-\lambda^{n+1} - 1 + 2\lambda^n = 0. \quad (15)$$

So, the polynomial λ^n- λ^{n-1} - λ^{n-2} -...- λ -1 = 0 has the same real roots as $-\lambda^{n+1} - 1 + 2\lambda^n = 0$. We find it easier to consider (15), so

$$\lambda = 2 - \frac{1}{\lambda^n}.$$

Now, $\lambda_{max} > 1$, so $(\lambda_{max})^n \to \infty$ thus

$$\lim_{n\to\infty} 2 - \frac{1}{(\lambda_{max})^n}$$
$$= 2 - 0$$
$$= 2.$$

Thus, as n increases, the dominant eigenvalue of the ratio $\frac{x(n+1)}{x(n)}$ for the "n"bonacci sequence converges to 2.

An alternate proof of this fact can be obtained by considering the maximum-modulus root of the polynomial of the form

$$\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_2\lambda^2 + a_1\lambda + a_0 = 0. \quad (16)$$

It can be shown, [3], that the maximum-modulus root of (16) has a generalized continued fraction representation given by

$$\lambda_{max} = \lim_{n\to\infty} \left(-a_{n-k} + \frac{a_{n-k-1}}{a_{n-1}}\right). \quad (17)$$

For the polynomial

$$\lambda^n - \lambda^{n-1} - \lambda^{n-2} - \ldots - \lambda - 1 = 0$$

for the n-bonacci relation, we have (16) with

$$a_{n-1} = a_{n-2} = \ldots = a_1 = a_0 = -1.$$

Thus,

$$\lambda_{\max} = \lim_{n\to\infty} \left(-(-1) + \frac{-1}{-1}\right) = 2.$$

The author acknowledes Dr. John Spraker for pointing out the relation (17).

It is also interesting to note that the realtion

$$\lambda = 2 - \frac{1}{\lambda^n}$$

gives a continued fractions representation different from (17). In fact,

$$\begin{aligned}\lambda &= 2 - \frac{1}{\lambda^n} \\ &= 2 - \frac{1}{(2 - \frac{1}{\lambda^n})^n} \\ &= 2 - \frac{1}{\left(2 - \frac{1}{(2 - \frac{1}{\lambda^n})^n}\right)^n} \\ &= 2 - \frac{1}{\left(2 - \left(\frac{1}{(\ldots)^n}\right)^n\right)}.\end{aligned}$$

The question is an open one relative to how the convergents of this continued fraction for each n compare to the dominant eigenvalue of the n-bonacci sequence.

Chapter Three

Fibonacci Sequence Gone Random

■ 3.1 A Randomized Classic Fibonacci Sequence

The Fibonacci numbers, as we have seen from the previous work, are quite remarkable. Not only do they occur often within our environment but the previous chapter showed that they are also naturally connected with the golden ratio $\frac{1+\sqrt{5}}{2} \approx 1.61$. What would happen to this sequence of numbers if an element of randomness were introduced? Would the ratio of terms of the sequence still converge, and if so to what?

Computer scientist Divakar Viswanath, who is currently a Dickson Instructor at the University of Chicago, explored this exact topic. Viswanath considered a random Fibonacci sequence defined by

$$x(0) = 1,$$

$$x(1) = 1,$$

$$x(n) = \pm\, x(n-1) \pm x(n-2),$$

with the ± signs being independent of each other and each term is either being added or subtracted based on a probability of $\frac{1}{2}$ [6].

Viswanath's research began with a simple observation. We know that the classic Fibonacci sequence increases exponentially as n increases. This fact prompted the question whether or not this random Fibonacci sequence that he defined would do the same, and if so, at what rate. To discover the answer meant introducing the Stern-Brocot division of the real line, random matrix products, fractals, and a computer calculation [8].

Through his work, Viswanath discovered that the random Fibonacci sequence did increase exponentially like its classic deterministic counterpart. And the rate at which it increases, its dominant eigenvalue (terminology is that of Viswanath and we will use it as well) is 1.13198824... [8]. This random Fibonacci sequencing has been justly dubbed the "Vibonacci Numbers" by a recent article in the *American Scientist* [1].

■ 3.2 Different Probabilities for the Random Fibonacci Sequence

Viswanath only considered the case in which each term was either added or subtracted based upon a probability of $\frac{1}{2}$. What happens when we consider different probabilities? Will the ratio converge? Specifically, consider the sequence

$$x(n) = z_1 x(n-1) + z_2 x(n-2) \tag{18}$$

with the z_i's, i = 1,2, being independent random variables such that

$$z_i = \begin{cases} +1 & \text{with probability } p \\ -1 & \text{with probability } q \end{cases}$$

where $p + q = 1$.

For instance, consider the sequence generated under a probability of $p = \frac{1}{4}$, so $q = \frac{3}{4}$. This sequence is such that the subtraction of the terms is more favorable than the addition. Some examples of these sequences with $p = \frac{1}{4}$ are as follows:

1,1,0,1,-1,-2,3,-5,2,...

1,1,-2,3,1,-4,5,-9,14,...

Now, let us consider another example of sequences generated using the probabilities of $p = \frac{1}{3}$ and $q = \frac{2}{3}$. Once again, it is more favorable to subtract the two terms rather than adding them. These conditions give, as examples, the following possible sequences:

1,1,2,1,-3,4,7,3,-10,...

1,1,0,1,1,-2,3,1,-4,...

Now, Viswanath considered the random sequence obtained when $p = \frac{1}{2}$ and $q = \frac{1}{2}$. Thus, neither the addition nor the subtraction of the two terms is said to be dominant, for they both should occur at the same rate. A few possible sequences are

1,1,-2,3,-1,-4,-3,1,-2,...

1,1,0,-1,-1,-2,1,-1,-2,...

The classic Fibonacci sequence always appears in the same order, for it is a deterministic sequence. However, we observe that there are many possible sequences that can and do occur based upon the randomness that has been introduced.

It can be noted, from the sequences listed above, that the random Fibonacci sequences are obtained through a technique similar to that used in obtaining the classic Fibonacci sequence. We know that the previous two terms are being added together to create the next term in the sequence, and under the classic Fibonacci case those two terms are always positive. When an element of randomness is introduced, instead of always adding two positive terms it is left up to chance whether we add two negative terms, one negative and one positive, or two positive terms.

So, since the classic and random Fibonacci sequences follow a similar pattern, can one suggest that the ratio of consecutive terms of the random sequences converge to some constant as the classic case does? Viswanath's research proved that in the case of $p = \frac{1}{2}$ and $q = \frac{1}{2}$, the ratio $\frac{x(n+1)}{x(n)}$ converges to approximately 1.13 [8]. But, how does the ratio $\frac{x(n+1)}{x(n)}$ react when presented with cases such that $p \neq \frac{1}{2}$? To further examine this point, let us consider the following *Mathematica* code:

```
Needs["DiscreteMath`Combinatorica`"]
SeedRandom[];
```

```
randomfib[k_, p_, n_, s_] := ( L = KSubsets[Range[n], s];
  For[z = 1, z <= Length[L], z++,
   fib = Table[1, {j, 1, n}];
 {For[i = 1 , i < k, i++,
   {newfib = 0;
    rv = Table[If[p > Random[], 1, -1], {j, 1, n}];
                s
      newfib =  Σ  rv[[L[[z]][[j]]]] fib[[L[[z]][[j]]]];
               j=1
       Do[fib[[n - j]] = fib[[n - j - 1]], {j, 0, n - 1}];
fib[[1]] = newfib }];
    Print[{L[[z]], N[Abs[fib[[1]]]^(1/k) ]}]};]; )
```

This code needs the input values of k, p, n, and s, where k is the total number of terms being generated in the sequence, p is the probability by which the terms are being added, n is the number of terms in consideration when forming the sequence (for example, in the classic Fibonacci case, n = 2), and s represents the number of terms considered for a subset of n terms. If s = n, then we are evaluating the "n"bonacci case. The ability to generate subsets will prove beneficial for later analysis.

Given these values, the command KSubsets yields all the possible subsets of n terms that have a length of s. The first for loop controls the program, allowing for each subset to be calculated. The inner for loop executes k times creating a random sequence consisting of k terms. The end result is the terms considered in each subset and its approximate dominant eigenvalue.

The following command executes the above code and considers 50,000 terms of the random Fibonacci sequence with $p = \frac{1}{2}$.

```
y = randomfib[50000, .5, 2, 2]
```

{{1,2},1.1345}

As Viswanath's research showed, this sequence converges approximately to 1.13. Now, consider the same command, except this time considering $p = \frac{1}{4}$, followed by $p = \frac{1}{3}$.

```
randomfib[50000, .25, 2, 2]
```

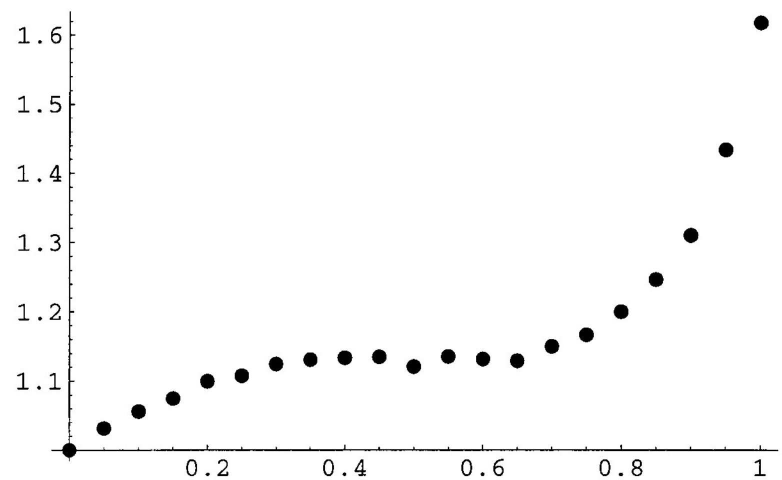
{{1,2},1.1062}

```
randomfib[50000, 1 / 3, 2, 2]
```

{{1,2},1.12533}

When the Fibonacci case is randomized using a probability of $p = \frac{1}{4}$, the ratio $\frac{x(n+1)}{x(n)}$ converges to the approximate dominant eigenvalue 1.10. Similarly, when $p = \frac{1}{3}$, $\frac{x(n+1)}{x(n)}$ has approximate dominant eigenvalue 1.12.

It appears that as p increases in value, so does the dominant eigenvalue. This increase follows from the increased probability favoring the addition of terms. Consider the graph below, which plots increasing p values and their corresponding dominant eigenvalues for the random Fibonacci case.



-Graphics-

This confirms, at least graphically, our thoughts that the dominant eigenvalue increases as p increases. Note as $p \to 1$, the dominant eigenvalue goes to the golden ratio, as we expect.

■ 3.3 A Randomized Tribonacci case

So far, we have considered only the random Fibonacci case. What would result if we were to apply this same type of random element to the Tribonacci case; that is, a random sequence defined by

$$x(0) = 1,$$

$$x(1) = 1,$$

$$x(2) = 1,$$

$$x(n) = z_1 x(n-1) + z_2 x(n-2) + z_3 x(n-3),$$

with the z_i's, i = 1,2,3, being independent random variables such that

$$z_i = \begin{cases} +1 & \text{with probability } p \\ -1 & \text{with probability } q \end{cases}$$

where $p + q = 1$.

Thus, the same technique applied to the Fibonacci case is now being implemented for the Tribonacci case. Using the Tribonacci sequence method and a probability of $p = \frac{1}{2}$ and $q = \frac{1}{2}$, the following possible sequences, for example, are produced:

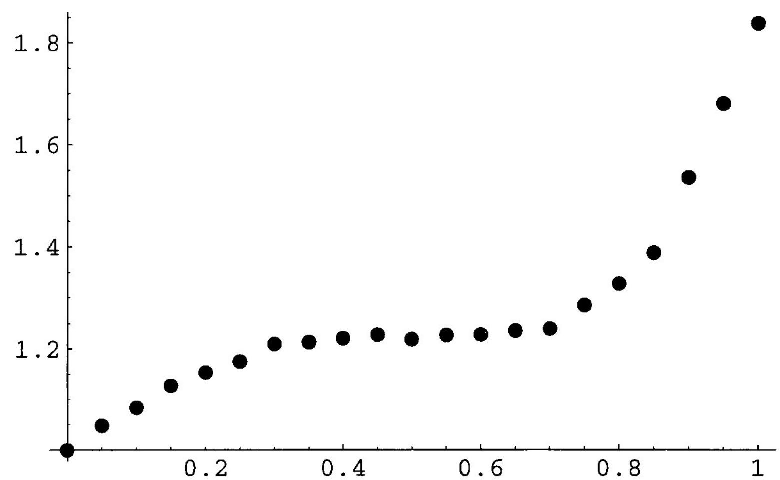
1,1,1,1,-3,-5,-7,9,-21,5,...

1,1,1,-3,-5,9,-1,15,7,23,...

As noted in the random Fibonacci case, there are many possible sequences that may occur due to the element of randomness applied to the Tribonacci sequence. Since the random Tribonacci case is obtained in the same manner as the random Fibonacci case, we can suggest that the ratio $\frac{x(n+1)}{x(n)}$ converges to some constant for each value of p. The code used earlier in the Fibonacci case can be executed here as well to determine the approximate dominant eigenvalue.

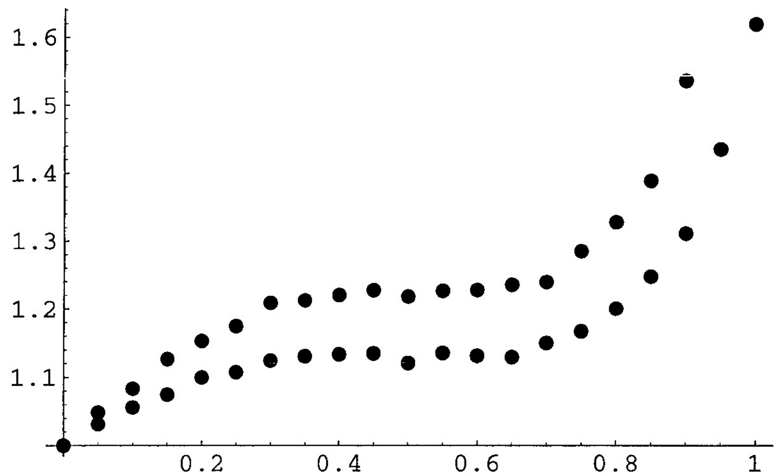
```
randomfib[50000, .5, 3, 3]
{{1,2,3},1.22163}
```

In the random Tribonacci case, $p = \frac{1}{2}$ has approximate dominant eigenvalue 1.22. If we consider different values of p, will the dominant eigenvalue increase as p increases, as in the random Fibonacci case? This conjecture does follow, as seen in the graph below.



-Graphics-

Again, notice as p→1, the dominant eigenvalue goes to 1.83, which is the dominant eigenvalue of the (deterministic) Tribonacci sequence. It is also interesting to note the shape of each of these two previous graphs. Indeed the shape is preserved for n=2 and n=3.



■ 3.4 Some Randomized Subsequences

We have just seen that in the random Fibonacci case, the ratio $\frac{x(n+1)}{x(n)}$ converges to the appropriate dominant eigenvalue depending upon the given values of p and q. The same holds true for the random Tribonacci case as well. And there is a direct correlation between the value of p and the dominant eigenvalue. Does this random sequencing method yield comparative eigenvalues when applied to other cases?

As before, with the regular Fibonacci sequencing method, let us evaluate the random Tribonacci case and all of its subcases. We know there are only three subcases using this technique and they are

$$x(n) = z_1 x(n-1) + z_2 x(n-2), \tag{19}$$

$$x(n) = z_1 x(n-1) + z_2 x(n-3), \tag{20}$$

$$x(n) = z_1 x(n-2) + z_2 x(n-3). \tag{21}$$

We have already examined equation (19), which is the recurrence relation for the random Fibonacci sequence that has an approximate dominant eigenvalue of 1.13. The approximate dominant eigenvalues for equations (20) and (21) can be found by implementing the *Mathematica* code used earlier on the random Fibonacci sequences.

Thus,

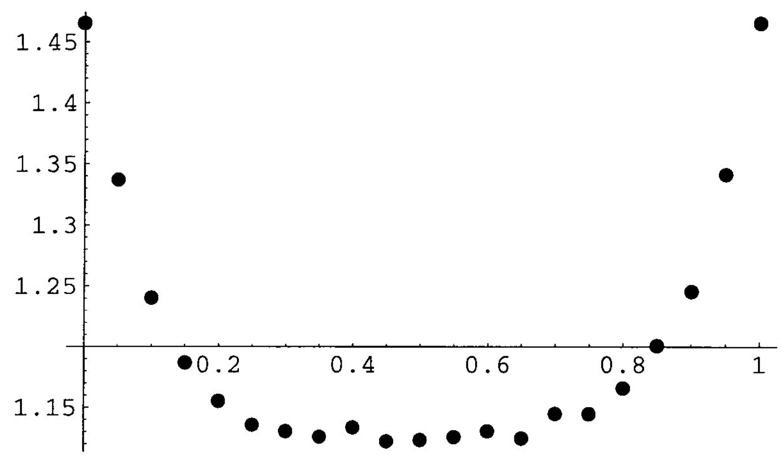
```
randomfib[50000, .5, 3, 2]
{{1,2},1.13395}
{{1,3},1.12683}
{{2,3},1.10691}
```

This output tells us that the dominant eigenvalue for equation (20) is approximately 1.12 and that for equation (21) is approximately 1.10. As before, notice that

$$1.10 < 1.12 < 1.13.$$

Thus the inequality among the dominant eigenvalues is preserved even with the presence of randomness.

As before, let us consider the effect on the dominant eigenvalues, for each subcase, as p increases. We have already observed what happens in the first case, for it is the same representation as the random Fibonacci case. So, let's examine what happens when observing the cases that appear in equations (20) and (21), respectively.



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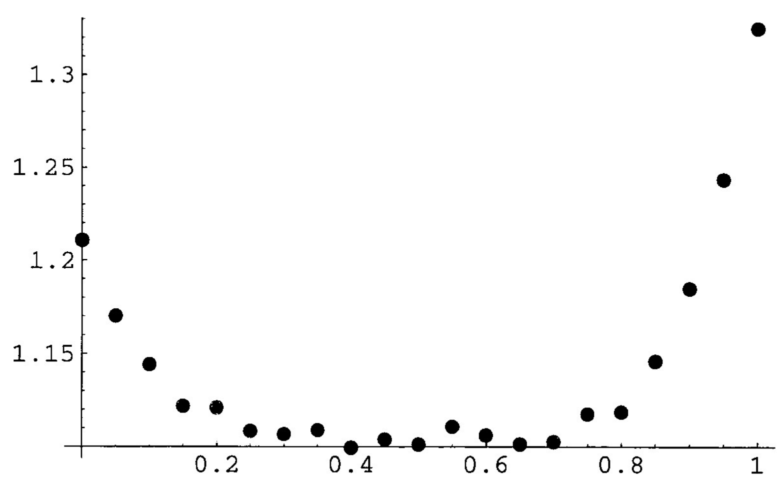

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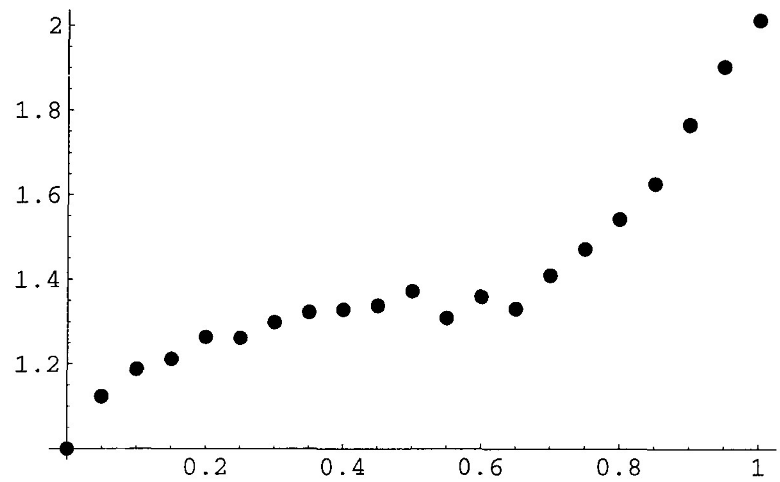
These graphs plot the increase of p, which signifies the probability of adding terms, and its corresponding dominant eigenvalues. The dip in the graph is evident from the fact that q is much larger than p, thus the subtraction of terms is more likely to occur, creating a lower dominant eigenvalue to be present.

■ 3.5 The Dominant Eigenvalue in the Randomized Limiting Case

We observed in the deterministic case that the dominant eigenvalues for which each sequence converges increases as n increases. Our work for the random cases, thus far, upholds this aspect. For, we have found the random Fibonacci case has dominant eigenvalue 1.13 and the random Tribonacci case has dominant eigenvalue of 1.22, so

$$1.13 < 1.22.$$

We can pose the question does the ratio of random "n"bonacci sequences, $n \geq 2$, converge to some constant as n increases? The graph below examines the dominant eigenvalue as p increases, for $n = 100$ terms in consideration.



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This suggests, graphically, that the random "n"bonacci sequence converges to 2, as p increases.

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